Laminar Free-Shear Layers in Power-Law Fluids

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This discussion of an analytical treatment of the steady laminar free-shear layer between two parallel streams of power-law fluid extends the work of Lock and others (*1* to *3*) dealing with Newtonian fluids.

We consider incompressible two-dimensional momentum exchange between a stream of power-law fluid with velocity U_1, density ρ_1, and rheological parameters m_1 and n_1 and a parallel stream with velocity U_2, density ρ_2, and rheological parameters m_2 and n_2. We assume isothermal flow with negligible mass diffusion. The axis of x is horizontal, in the direction of motion of the free streams, and the positive y axis points toward the higher velocity stream. The origin is taken as the point at which the two fluids are supposed to come into contact first. The velocity components in the x and y directions are u and v, respectively. The interface between the two fluids is designated by $y^o(x)$. Making the usual boundary-layer assumptions and noting that the pressure is everywhere constant, one finds that the momentum equations for the two fluids are (*4*)

$$\rho_1\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)= m_1\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^{n_1}$$

higher velocity fluid (1)

$$\rho_2\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)= m_2\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)^{n_2}$$

lower velocity fluid (2)

Since the velocity field must satisfy the free-stream conditions and the velocity and stress must be continuous at the interface separating the two fluids, the boundary conditions are

$$u\rightarrow U_1 \text{ as } y\rightarrow +\infty \tag{3}$$

$$u\rightarrow U_2 \text{ as } y\rightarrow -\infty \tag{4}$$

$$\left.\begin{aligned}u_1(y^o)&=u_2(y^o)\\ v_1(y^o)&=v_2(y^o)\end{aligned}\right. \tag{5}$$

$$m_1\left[\left(\frac{\partial u_1}{\partial y}\right)^{n_1}\right]_{y^o}=m_2\left[\left(\frac{\partial u_2}{\partial y}\right)^{n_2}\right]_{y^o} \tag{6}$$

In the higher velocity fluid we look for a similarity solution for the stream function ψ of the form

$$\psi=U_1\left(\frac{m_1x}{\rho_1U_1^{2-n_1}}\right)^{1/(n_1+1)}f_1(\eta_1{}^*) \tag{7}$$

where $\eta_1{}^*=\eta_1-\eta_1{}^o$ and

$$\eta_1=y\left(\frac{\rho_1U_1^{2-n_1}}{m_1x}\right)^{1/(n_1+1)} \tag{8}$$

For the lower velocity stream, ρ_1, m_1, n_1 and f_1 are replaced by ρ_2, m_2, n_2 and f_2 in Equations (7) and (8). Equations (1) and (2) then become

$$n_1(n_1+1)f_1'''+f_1(f_1'')^{2-n_1}=0 \tag{9}$$

$$n_2(n_2+1)f_2'''+f_2(f_2'')^{2-n_2}=0. \tag{10}$$

The free-stream and interface boundary conditions become

$$f_1'(\infty)=1 \tag{11}$$

$$f_2'(-\infty)=\lambda \tag{12}$$

$$f_1'(0)=f_2'(0) \tag{13}$$

$$\rho_1\left(\frac{\rho_1U_1^{2-n_1}x^{n_1}}{m_1}\right)^{-1/(n_1+1)}[f_1''(0)]^{n_1}=\rho_2\left(\frac{\rho_2U_2^{2-n_2}x^{n_2}}{m_2}\right)^{-1/(n_2+1)}[f_2''(0)]^{n_2} \tag{14}$$

$$f_1(0)=f_2(0)=0 \tag{15}$$

Equations (11) to (14) are equivalent to Equations (3) to (6), respectively, and Equation (15) labels the interface as the zero streamline.

From Equation (14) we find that the shear-stress boundary condition becomes similar only for $n_1 = n_2$. Hence, no similarity solution exists if the two fluids have different values of the rheological parameter n, and we shall consider here only the case $n_1 = n_2 = n$. In this case the stress boundary condition reduces to

$$f_1''(0)=(\rho\, m^{1/n})^{1/(n+1)}f_2''(0) \tag{16}$$

where $\rho=\rho_2/\rho_1$ and $m=m_2/m_1$.

As noted in reference *4*, when $n \geqq 2$, we cannot find a function which will satisfy either Equation (9) or (10) and the corresponding free-stream condition $f'(\pm\infty)=$ const.; consequently the free-stream conditions must be changed into a more general form. Although the same procedure can easily be adopted here, available experiments indicate no values of n as high as 2 and so such calculations would appear to have no practical significance.

The solutions for the dimensionless velocity profiles depend only on the ratio λ of the velocities of the two streams, on the power index n of the fluids, and on the

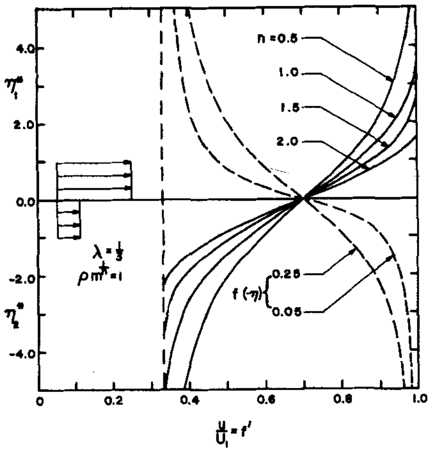

Fig. 1. Dimensionless velocity profiles for laminar free-shear layers in power law fluids; $\rho m^{1/n} = 1$, $\lambda = 1/3$. Profiles for $n = 0.05$ and 0.25 are inverted for clarity.

Table 1. Results of Numerical Integration and Interface Deflection η_1^o (additional Condition $m/\rho = 1$) for $\rho m^{1/n} = 1$

λ	n	$f_1'(0)$	$f_1''(0)$	α	β	η_1^o
0	0.05	0.599	0.532	0.441	1.982	−0.441
	0.25	0.594	0.182	0.861	2.796	−0.861
	0.50	0.592	0.165	0.764	2.229	−0.764
	1.00	0.587	0.200	0.529	1.238	−0.529
	1.50	0.584	0.245	0.408	0.867	−0.408
	2.00	0.582	0.288	0.337	0.690	−0.337
1/3	0.05	0.700	0.333	0.379	0.727	−0.56
	0.25	0.699	0.121	0.711	1.236	−1.01
	0.50	0.698	0.117	0.600	0.981	−0.835
	1.00	0.697	0.154	0.375	0.567	−0.508
	1.50	0.696	0.198	0.271	0.393	−0.362
	2.00	0.695	0.241	0.215	0.311	−0.289

parameter $\rho m^{1/n}$. For $\lambda = 0$ and $\rho m^{1/n} \to \infty$, the slower fluid is stationary and the problem reduces to that of the boundary layer on a stationary surface considered in reference *4*. For $\rho m^{1/n} \to \infty$ and $\lambda \neq 0$, we have the case of a boundary layer on a moving solid surface.

The interface position η_1^o is as yet undetermined and it is necessary to specify one additional boundary condition in order to obtain a unique solution. The appropriate boundary condition for this problem in Newtonian fluids can be shown to be (*2, 3*)

$$U_1 V_1 + \left(\frac{\rho_2 \mu_2}{\rho_1 \mu_1}\right)^{1/2} U_2 V_2 = 0$$

where V_1 and V_2 are the y components of the velocity at the edges of the mixing region. For power-law fluids, the analysis is readily generalized to give the boundary condition

$$U_1 V_1 + (\rho m^{1/n})^{n/(n+1)}\, U_2 V_2 = 0 \tag{17}$$

From Equation (17), the interface position is then found to be

$$\eta_1^o = -\frac{\alpha + (\rho m^{1/n})^{n/(n+1)}\,(m/\rho)^{1/(n+1)}\,\lambda\,\beta}{1 + (\rho m^{1/n})^{n/(n+1)}\,\lambda^2} \tag{18}$$

where α and β are determined by the asymptotic form of the solutions for large η and must be obtained from numerical solutions (*2*).* Since α and β are positive and $0 \leq \lambda \leq 1$, the interface always deflects toward the slower moving stream. To calculate the interface position, it is necessary to specify the additional parameter m/ρ.

Equations (9) and (10) with boundary conditions (11) to (16) were solved numerically by choosing values for the interface velocity and shear stress and integrating in both directions until the velocities become constant. The values of interface velocity and shear stress were iterated until the desired free-stream velocity ratio was obtained.

Figures 1 and 2 illustrate the behavior of the solutions for $\lambda = 0$ and 1/3, and the salient numerical results are presented in Tables 1 and 2. Profiles for several values of n, with $\rho m^{1/n} = 1$ and $\lambda = 1/3$ are given in Figure 1. A minimum in $f''(0)$, the slope of the velocity profile at the interface, occurs near $n = 0.37$. For $n > 0.37$, the effective width of the shear layer decreases and the rate of variation of velocity in the streamwise direction increases as n increases. For $n < 0.37$, the shear layer increases in width and the velocity varies more slowly with x as n increases. For dilatant fluids, the velocity everywhere approaches the free-stream values more rapidly than for Newtonian fluids, confining the effects of mixing to a thinner region as n increases. For pseudoplastic fluids, the velocity asymptotically approaches free-stream conditions more slowly than for Newtonian fluids, the approach becoming slower as n decreases. Because of the reversal in behavior noted at $n = 0.37$, the effective width of the mixing zone, which is larger than for Newtonian fluids, at first increases and then decreases as n becomes smaller. For $\lambda = 0$ and 1/3, Table 1 gives the corresponding interface velocities, values of α and β, interface deflections ($m/\rho = 1$), and also $f''(0)$ which may be used to calculate the shear stress at the interface. The interface velocity does not depend on n explicitly, and the variation of $f''(0)$ with λ ($\rho m^{1/n} = 1$) and the variation of $f'(0)$ with $\rho m^{1/n}$ for $\lambda = 0$ are the same as given by Lock (*1*) for Newtonian fluids when μ is replaced by $m^{1/n}$. The interface deflection is roughly proportional to the effective width of the shear layer.

Results with $\rho m^{1/n} \neq 1$ for several combinations of pseudoplastic fluids of experimental interest are given in Figure 2 and Table 2. In each case the two fluids have (nearly) the same density and value of n (*6, 7*). Comparing the two cases in Figure 2 with $n = 0.56$, we find that changes in m produce much larger variations in interface velocity than are caused by differences in μ of the

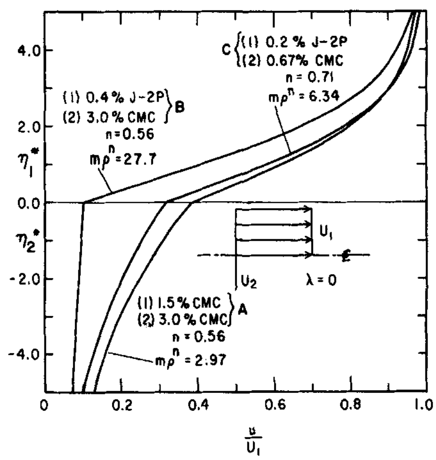

Fig. 2. Dimensionless velocity profiles for several pseudoplastic free-shear layers; $\rho m^{1/n} \neq 1$, $\lambda = 0$.

* Some errors in Yen's work have been pointed out by Baker (*5*).

TABLE 2. INTERFACE DEFLECTION AND RESULTS OF NUMERICAL INTEGRATIONS FOR EXAMPLES IN FIGURE 2

λ	Fluids	$f_1'(0)$	$f_1''(0)$	α	β	η_1^o
0	A	0.381	0.250	1.079	1.934	−1.079
	B	0.101	0.319	1.660	0.940	−1.660
	C	0.317	0.272	1.114	1.416	−1.114
1/3	A	0.546	0.186	0.802	0.678	−1.41
	B	0.375	0.252	1.090	0.234	−3.38
	C	0.503	0.211	0.787	0.452	−1.57

same order with Newtonian liquids of equal densities, but the effect on interface deflection is roughly the same as for Newtonian fluids. For this example, increasing m_1 by a factor of 9.3 increases the interface velocity by a factor of 3.8 and increases the interface deflection y^o at any axial distance x by a factor of 2.7. To produce the same changes in interface velocity and deflection in Newtonian fluids of the same density, with $\mu = m^{1/n}$ initially, it is necessary to increase μ_1 by factors of 54 and 3.7, respectively.

One would expect comparison of these shear-layer solutions with experiment to show better agreement than is found in the case of jets (*8*), because the power-law approximation is best in the center of the shear layer where the shear rate is largest, whereas it is known to be invalid on the centerline of a jet where the velocity gradient vanishes.

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NOTATION

f = dimensionless stream function
m, n = parameters characterizing a power law fluid
u, v = fluid velocity in x and y directions, respectively
U_1 = free-stream velocity of higher velocity fluid
U_2 = free-stream velocity of lower velocity fluid
x = distance from initial point of contact of two streams
y = distance from center line

Greek Letters

η = dimensionless similarity variable
λ = U_2/U_1 = ratio of free-stream velocities
ψ = stream function
ρ = fluid density
μ = Newtonian viscosity

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The Equation of Convective Diffusion and Its Solution in the Small Penetration Approximation

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Because the diffusion coefficient in a liquid is very small, the depth of penetration by diffusion is also very small. It is possible therefore to use in numerous cases the small penetration approximation, that is, to solve the problems of mass transfer by means of the solutions valid for a semiinfinite liquid (*1, 2*). The aim of the present communication is to examine in the framework of this approximation the mass transfer through a free-moving interface. The equation of convective diffusion for a liquid having a free-moving interface may be solved exactly by means of a similarity transformation, as is illustrated by the problem of the absorption of a gas in a thin liquid film in wave motion along a vertical wall.

The equation for a moving interface between a liquid and a gas is, with respect to a fixed-coordinate system xOy,

$$y = H\,(x, t) \tag{1}$$

Since we want to know the mass flux at the free interface, it is natural to write the convective-diffusion equation in a frame of reference bound to this interface. Such a coordinates system may be defined, for instance, by the distance y_1 to the interface and by the corresponding distance x_1 along the interface.

When the depth of penetration by diffusion is small, we may use for the velocity components in the convective-diffusion equation expressions valid in the vicinity of the interface:

$$u_1 \approx (u_1)_{y_1=0} = u_{10} \tag{2}$$

$$v_1 \approx (v_1)_{y_1=0} + \left(\frac{\partial v_1}{\partial y_1}\right)_{y_1=0} \cdot y_1 = \left(\frac{\partial v_1}{\partial y_1}\right)_{y_1=0} y_1 \tag{3}$$

($(v_1)_{y_1=0} = 0$ since the frame of reference is bound to the interface, which is a streamline.)

Approximating also $\nabla^2 c \approx (\partial^2 c/\partial y_1^2)$ and taking into account that in the region of interest $y_1 << R$, the equation of convective diffusion in the small penetration ap-